ECE 343: Signals and Systems

Final

1. When analyzing frequency modulation schemes, signals of the form $x(t) = \cos(\beta\cos(2\pi\alpha t))$ are frequently encountered.

   (a) $x$ is periodic. Find the (smallest) period.

   (b) Given that $e^{jw\cos(\phi)} = \sum_{n=-\infty}^{\infty} j^n J_n(w) e^{jn\phi}$, where $J_n(w)$ is a Bessel function of the first kind[1], find the Fourier series of $x$.

2. Eigenvectors.

   (a) What does it mean for something to be an eigenvector? An eigenvalue? What are the eigenfunctions of continuous time LTI systems?

   (b) The Fourier transform takes a signal as input and outputs another signal. Is the FT a system? Is it an LTI system? If it is not LTI, specify whether it fails linearity, time-invariance, or both.

   (c) Find the Fourier transform of $x_1(t) = te^{-\pi t^2}$.

   (d) Is $x_1$ an eigenfunction of the Fourier transform? If so, what is the eigenvalue?

   (e) Find the Fourier transform of $x_2(t) = te^{-t^2}$.

   (f) Is $x_2$ an eigenfunction of the Fourier transform? If so, what is the eigenvalue?

3. Consider the LTI system governed by the differential equation $\frac{d^2}{dt^2}y + 6\frac{d}{dt}y + 5y = 4\frac{d}{dt}x$.

   (a) Find the impulse response of the system.

   (b) Find the transfer function of the system.

   (c) Is this system BIBO stable? Is it causal? Justify your answers.

   (d) Find the output of the system if the input is $e^{-2t}$.

   (e) Find the output of the system if the input is $e^{-2t}u(t)$.

4. Let $x_1[n] = 2^{-n}\sin(\frac{\pi}{3}n)u[n]$ and $x_2[n] = 2^{-n}\sin(-\frac{5\pi}{3}n)u[n]$.

   (a) Find the Z-transform of $x_1[n]$.

   (b) Find the Z-transform of $x_2[n]$.

   (c) Explain why $X_1(z) = X_2(z)$.

5. (Bonus) Let $x(t) = p(t)u(t)$, where $p(t)$ is a $T_0$-periodic sawtooth wave.

$$p(t) = \begin{cases} t & 0 < t < T_0 \\ p(t+T_0) & \text{else} \end{cases}$$

   Find the output of the system with the transfer function $H(s) = \frac{1}{(s+1)}$ when $x(t)$ is the input.

[1] No, you don't need to know what a Bessel function (of any kind) is. They look somewhat like sinc pulses, but go like $\frac{1}{\sqrt{t}}$ instead of $\frac{1}{t}$ for large $t$.

## 1.a

$\cos(2\pi\alpha t)$ is $\frac{1}{\alpha}$-periodic. Therefore $x(t)$ is $\boxed{\frac{1}{\alpha}\text{-periodic}}$

$x(t-\frac{1}{\alpha}) =$

$\cos(\beta\cos(2\pi\alpha(t-\frac{1}{\alpha}))) = \cos(\beta\cos(2\pi\alpha t - 2\pi)) = \cos(\beta\cos(2\pi\alpha t)) = x(t)$

## 1.b

$$e^{jw\cos(\phi)} = \sum_{n=-\infty}^{\infty} j^n J_n(w) e^{jn\phi}$$

Let $\tilde{x}(t) = e^{j\beta\cos(2\pi\alpha t)}$. Then $x(t) = 2\,\mathrm{Re}(\tilde{x}(t))$.

$$x(t) = 2\,\mathrm{Re}\left(e^{j\beta\cos(2\pi\alpha t)}\right) = 2\,\mathrm{Re}\left(\sum_{n=-\infty}^{\infty} j^n J_n(w) e^{jn2\pi\alpha t}\right)$$

$$= \sum_{k=-\infty}^{\infty} 2\,\mathrm{Re}\left(j^k J_k(w) e^{2\pi j\alpha k t}\right)$$

$$= \sum_{k=-\infty}^{\infty} j^k J_k(w) e^{2\pi j\alpha k t} + \left(j^k J_k(w) e^{2\pi j\alpha k t}\right)^*$$

$$= \sum_{k=-\infty}^{\infty} j^k J_k(w) e^{2\pi j\alpha k t} + (-j)^k J_k^*(w) e^{-2\pi j\alpha k t}$$

$$= \sum_{k=-\infty}^{\infty} j^k J_k(w) e^{2\pi j\alpha k t} + \sum_{k=-\infty}^{\infty} j^{-k} J_k^*(w) e^{-2\pi j\alpha k t} \qquad \left(\frac{1}{j} = -j\right)$$

$$= \sum_{k=-\infty}^{\infty} j^k J_k(w) e^{2\pi j\alpha k t} + \sum_{k=-\infty}^{\infty} j^k J_{-k}^*(w) e^{2\pi j\alpha k t} \qquad (k \to -k)$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{j^k\left(J_k(w) + J_{-k}^*(w)\right)}_{X[k]} e^{2\pi j\alpha k t}$$

$\Leftarrow$ this is a Fourier series and it is equal to $x(t)$

## 2.a

An eigenvector of a linear transformation is a vector that is only scaled by the transformation. The result of the transformation is exactly the same as the input, only it is multiplied by a scalar constant. That constant is called the eigenvalue.

$$x \rightarrow \lambda x$$

The eigenvectors of LTI systems are exponentials:

CT, aperiodic: $e^{st}$ $\quad s \in \mathbb{C}$

CT, periodic: $e^{2\pi j k t/T_0}$

DT, aperiodic: $z^n$ $\quad z \in \mathbb{C}$

DT, periodic: $e^{2\pi j k n/N_0}$

## 2.b

Systems take one or more signals and output one or more signals. Therefore <u>the FT is a system.</u>

$\boxed{\text{FT is Linear:}}$ $\mathcal{F}\{a x_1 + b x_2\} = a\mathcal{F}\{x_1\} + b\mathcal{F}\{x_2\}$

$\boxed{\text{FT is not TI:}}$ $\mathcal{F}\{x(t-\tau)\} = e^{-2\pi j f \tau}\mathcal{F}\{x(t)\} \neq \mathcal{F}\{x\}(f-\tau)$

## 2.c

$$\mathcal{F}\{e^{-\pi t^2}\} = e^{-\pi f^2}$$

$$\mathcal{F}\{te^{-\pi t^2}\} = \frac{-1}{2\pi j}\frac{d}{df}\left(e^{-\pi f^2}\right) = -\frac{1}{2\pi j}e^{-\pi f^2}(-2\pi f) = \boxed{\left(\frac{1}{j}\right)\left(fe^{-\pi f^2}\right)}$$

## 2.d

$x_1(t) \rightarrow \frac{1}{j}x_1(f)$ Ignore the fact that the independent variable changes. It is a dummy variable and is immaterial until we assign meaning to the abstract signal $x_1$. Therefore $x_1$ is an eigenfunction, with eigenvalue $\boxed{\frac{1}{j}}$

2.e

$$\mathcal{F}\{e^{-t^2}\} = \mathcal{F}\left\{e^{-\pi\left(\frac{t}{\sqrt{\pi}}\right)^2}\right\} = \sqrt{\pi}\, e^{-\pi(\sqrt{\pi} f)^2} = \sqrt{\pi}\, e^{-\pi^2 f^2}$$

$$\mathcal{F}\{t e^{-t^2}\} = \frac{-1}{2\pi j}\frac{d}{df}\left(\sqrt{\pi}\, e^{-\pi^2 f^2}\right) = \frac{-1}{2\pi j}\sqrt{\pi}\, e^{-\pi^2 f^2}\left(-\pi^2 2f\right)$$

$$= \boxed{-j\pi\sqrt{\pi}\, f e^{-\pi^2 f^2}}$$

2.f

$$x_2 \xrightarrow{\mathcal{F}} -j\pi\sqrt{\pi}\, f e^{-(\pi f)^2} = -j\sqrt{\pi}\, x_2(\pi f) \neq \lambda x_2(f)$$

the frequency scaling ruins the symmetry. It's not the same function multiplied by a scalar. <u>Not an eigenfunction.</u>

3.a)

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 5y = 4\frac{dx}{dt} \xrightarrow{\mathcal{L}} (s^2+6s+5)Y(s) = 4sX(s)$$

$$\frac{Y(s)}{X(s)} = \frac{4s}{s^2+6s+5} = H(s) = \frac{4s}{(s+1)(s+5)} = \frac{A}{s+1} + \frac{B}{s+5}$$

$$A = \lim_{s\to -1} \frac{4s}{s+5} = \frac{-4}{4} = -1$$

$$B = \lim_{s\to -5} \frac{4s}{s+1} = \frac{-20}{-4} = 5$$

$$H(s) = \frac{5}{s+5} - \frac{1}{s+1} \xrightarrow{\mathcal{L}^{-1}} \boxed{5e^{-5t}u(t) - e^{-t}u(t) = h(t)}$$

3.b)

From above, $\boxed{H(s) = \frac{4s}{(s+1)(s+5)} = \frac{5}{s+5} - \frac{1}{s+1}}$

3.c)

$h(t)$ is causal, therefore the system is causal

All poles are in LHP (left half-plane) and it's causal, so it is BIBO stable.

Also, $\int_{-\infty}^{\infty} |h(t)|\,dt = \int_0^{\infty} |5e^{-5t} - e^{-t}|\,dt \leq \int_0^{\infty} |5e^{-5t}| + |e^{-t}|\,dt$

$$= \int_0^{\infty} 5e^{-5t}\,dt + \int_0^{\infty} e^{-t}\,dt = \int_0^{\infty} e^{-u}\,du + \int_0^{\infty} e^{-t}\,dt = 2 < \infty$$

$u = 5t$

So $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty \rightarrow$ stable.

3.d

$e^{-2t}$ is an eigenfunction.

$$e^{-2t} \xrightarrow{sys} H(-2)e^{-2t} = \frac{4(-2)}{(3)(-1)} e^{-2t} = \boxed{\frac{8}{3} e^{-2t}}$$

3.e

$e^{-2t}u(t)$ isn't an eigenfunction. But we can use Laplace transforms. We could use Fourier transforms too, but they aren't as convenient.

$$\mathcal{L}\{e^{-2t}u(t)\} = \frac{1}{s+2} = X(s)$$

$$Y(s) = H(s)X(s) = \frac{4s}{(s+1)(s+5)(s+2)} = \frac{A}{s+1} + \frac{B}{s+5} + \frac{C}{s+2}$$

$$A = \lim_{s\to -1} \frac{4s}{(s+5)(s+2)} = \frac{-4}{(4)(1)} = -1$$

$$B = \lim_{s\to -5} \frac{4s}{(s+1)(s+2)} = \frac{-20}{(-4)(-3)} = -\frac{5}{3}$$

$$C = \lim_{s\to -2} \frac{4s}{(s+5)(s+1)} = \frac{-8}{(3)(-1)} = \frac{8}{3}$$

$$Y(s) = \frac{-1}{s+1} + -\frac{5}{3}\frac{1}{s+5} + \frac{8}{3}\frac{1}{s+2}$$

$$\boxed{y(t) = \left(-e^{-t} + -\frac{5}{3}e^{-5t} + \frac{8}{3}e^{-2t}\right)u(t)}$$

the eigen-response still appears

## 4.a

just use the table.

$$|\gamma|^n \sin(\beta n)\, u[n] \xrightarrow{z} \frac{z\,|\gamma| \sin(\beta)}{z^2 - 2|\gamma|\cos(\beta)\, z + |\gamma|^2}$$

$\gamma = 2^{-1}$

$|\gamma| = \gamma = 2^{-1}$

$\beta = \frac{\pi}{3}$

$$\boxed{X_1(z) = \frac{z\,\frac{1}{2}\sin\left(\frac{\pi}{3}\right)}{z^2 - \cos\left(\frac{\pi}{3}\right) z + \frac{1}{4}} = \frac{z\,\frac{\sqrt{3}}{4}}{z^2 - \frac{1}{2} z - \frac{1}{4}}}$$

## 4.b

$$X_2(z) = \frac{z\,\frac{1}{2}\sin\left(-\frac{5\pi}{3}\right)}{z^2 - \cos\left(-\frac{5\pi}{3}\right) z + \frac{1}{4}} = \boxed{\frac{\frac{\sqrt{3}}{4}\, z}{z^2 - \frac{1}{2} z + \frac{1}{4}}}$$

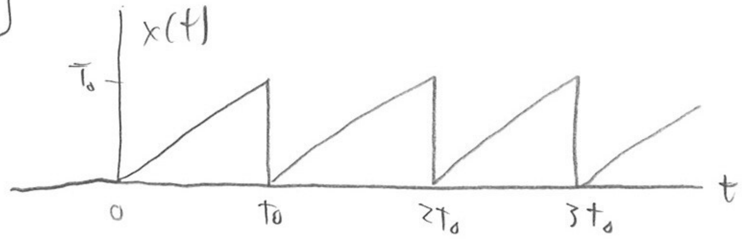
## 4.c

the simple answer is simply because $x_1 = x_2$.

$$\sin\left(\frac{\pi}{3} n\right) = \sin\left(\frac{\pi}{3} n - 2\pi n\right) = \sin\left(-\frac{5\pi}{3} n\right)$$

The deeper answer is that the discrete time frequency domain is periodic. $x_2$ is an alias of $x_1$. Even though the "frequency" of $x_2$ is different $\left(-\frac{5\pi}{3} \neq \frac{\pi}{3}\right)$, they are equivalent in discrete time.

5.)




set $g(t) = t(u(t) - u(t-T_0)) = tu(t) - (t-t_0)u(t-t_0) - T_0 u(t-T_0)$

$$G(s) = \frac{1}{s^2}\left(1 - e^{-st_0}\right) - T_0 e^{-sT_0}\frac{1}{s}$$

$$X(s) = \frac{G(s)}{1 - e^{-st_0}}$$

This isn't hard to derive if you remember Poisson Summation. See solutions for HW8.

$$Y(s) = H(s)\,X(s)$$

$$= \frac{1}{s+1}\,\frac{G(s)}{1-e^{-st_0}} = \frac{1}{s+1}\left(\frac{\frac{1}{s^2}\left(1-e^{-st_0}\right)}{1-e^{-st_0}} - \frac{t_0 e^{-sT_0}\frac{1}{s}}{1-e^{-st_0}}\right)$$

$$= \underbrace{\frac{1}{s^2(s+1)}}_{Y_t(s)\text{ transient}} + \underbrace{\frac{T_0 e^{-st_0}}{s(s+1)}\,\frac{1}{1-e^{st_0}}}_{Y_r(s)\text{ repeating}}$$

$$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$$B = \lim_{s\to 0} \frac{1}{s+1} = 1$$

$$C = \lim_{s\to -1} \frac{1}{s^2} = -1$$

Try $s=1$: $\frac{1}{1(1+1)} = \frac{1}{2} = \frac{A}{1} + \frac{1}{1} + \frac{-1}{2} \rightarrow A = \frac{1}{2} + \frac{1}{2} - 1 = 0$

$$\boxed{y_t(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s+1}\right\} = tu(t) - e^{-t}u(t)}$$

$\boxed{\text{5 cont.}}$

$$Y_r(s) = \frac{T_0 e^{-sT_0}}{s(s+1)} \frac{1}{1-e^{-sT_0}}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$A = \lim_{s \to 0} \frac{1}{s+1} = 1$$

$$B = \lim_{s \to -1} \frac{1}{s} = -1$$

$$Y_r(s) = \left(\frac{1}{s} - \frac{1}{s+1}\right) T_0 e^{-sT_0} \frac{1}{1-e^{-sT_0}}$$

$$\boxed{y_r(t) = \sum_{n=0}^{\infty} q(t - T_0 n), \quad q(t) = T_0\left(1 - e^{-(t-T_0)}\right)u(t-T_0)}$$

$$\boxed{y(t) = y_t(t) + y_r(t)}$$